Efficient kinetic Lattice-Boltzmann simulation for three-dimensional Hall MHD

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Introduction

Hall MHD is an extension of ideal MHD in which the velocities of ions and electrons are different because ions are decoupled from the magnetic field while electrons remain tied to it. In the framework of Hall MHD the generalized Ohm's law is modified by introducing the Hall electric field, a term proportional to $\boldsymbol{J}\times\boldsymbol{B}$. Exploring the Hall regime with direct numerical simulations (DNS) is a challenging task due to the large range of scales involved in the dynamics. In the present work, a novel lattice Boltzmann (LB) implementation is proposed as an alternative to classical approaches. It exploits the GPU capabilities to reach a profitable trade-off between accuracy of the simulated plasma dynamics and an optimal computational efficiency. We also propose a multi-GPU implementation which allows to reach very high-resolution keeping acceptable computational times. These features have encouraged us to deepen into the LB modeling and evaluate its practical utility for the study of Hall MHD turbulence.

Hall MHD equations

At the macroscopic level, the LB approach simulates the nearly incompressible and isothermal Hall MHD equations expressed as follows,

$$\partial_t\rho+\nabla\cdot(\rho\boldsymbol{u}) = 0$$
$$\partial_t(\rho\boldsymbol{u})+\nabla\cdot\left(\rho\boldsymbol{u}\otimes\boldsymbol{u}+pI+\frac{1}{2}|\boldsymbol{B}^2|I-\boldsymbol{B}\otimes\boldsymbol{B}\right) = \rho\nu\nabla^2\boldsymbol{u}$$
$$\partial_t\boldsymbol{B}+\nabla\cdot[(\boldsymbol{u}-\alpha_H\boldsymbol{J})\otimes\boldsymbol{B}-\boldsymbol{B}\otimes(\boldsymbol{u}-\alpha_H\boldsymbol{J})] = \eta\nabla^2\boldsymbol{B}$$
$$p = \rho c_s^2$$

Where ν and η are the kinematic viscosity and magnetic diffusivity, respectively.
The incompressibility is achieved in the limit of low Mach number $\mathrm{Ma}=|\mathbf{u}|/c_s\rightarrow 0$ with density fluctuations being $O(\mathrm{Ma}^2)$.
The Hall parameter α_H arises as a ratio between two length scales,

$$\alpha_H = \frac{V_A}{U_0L_0}\sqrt{\frac{m}{\mu_0ne^2}} = \frac{L_H}{L_0}$$

If $U_0=V_A$ (the Alfvèn velocity), then $L_H=d_i$ (ion skin depth). Hall effects are expected to prevail at scales smaller than L_H.

Hall MHD LB Method

In LBM, the flow complexity emerges from the repetition of simple of **collision** and **streaming** rules of collections of particle living on a regular lattice. The classical macroscopic quantities, such as density ρ, momentum $\rho\boldsymbol{u}$ and magnetic field **B** (and electric current density **J**) are obtained as the statistical moments of the distribution functions $f_i(\boldsymbol{x},t)$, for fluid, and $\boldsymbol{g}_i(\boldsymbol{x},t)$ for the magnetic field.

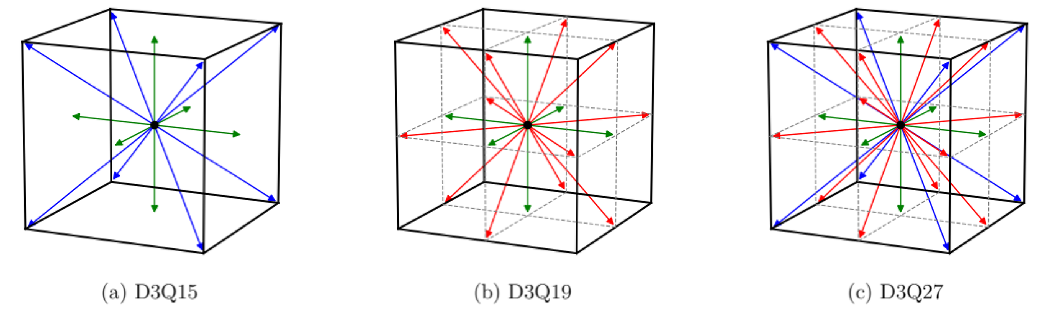


Fig. 1: Lattice structures commoly adopted for discretizing the velocity space.

LB approach for Hall MHD consists in two separate lattices: D3Q27 for the fluid and D3Q7 for the magnetic field, where D*d*Q*q* is the lattice discretization with dimension *d* and *q* microscopic velocities. The two lattices are coupled by the equilibrium distribution functions for the fluid,

$$f_i^{(0)} = f_i^{NS(0)}+\frac{w_i}{2c_S^4}\left[|\boldsymbol{B}^2||\boldsymbol{c}_i^2|-(\boldsymbol{c}_i\cdot\boldsymbol{B})^2\right]$$

and the *vector-valued* distribution functions for the magnetic field,

$$g_{i\alpha}^{(0)} = W_i\left[B_\alpha+\frac{1}{C^2}\xi_{i\beta}\left((u_\beta-\alpha_HJ_\beta)B_\alpha-B_\beta(u_\alpha-\alpha_HJ_\alpha)\right)\right]$$

The magnetic equilibrium functions account for the Hall term $-\alpha_H\boldsymbol{J}\times\boldsymbol{B}$. Importantly, the electric current density is implicitly obtained by the first-order moment of the non-equilibrium component of the magnetic field distributions ,

$$\boldsymbol{J}_{\boldsymbol{\gamma}} = -\frac{\omega_m}{C^2}\epsilon_{\alpha\beta\gamma}\boldsymbol{\Lambda}_{\boldsymbol{\alpha\beta}}^{\boldsymbol{neq}}+O(Ma^3)$$

where $\boldsymbol{\Lambda}_{\boldsymbol{\alpha\beta}}^{\boldsymbol{neq}}=\sum_i\xi_{i,\alpha}\otimes\left(g_{i,\beta}-g_{i,\beta}^{(0)}\right)$. Since the equilibrium function depends itself on the current density, we obtain $\boldsymbol{J}$ by inverting

$$\left(I+\frac{2\alpha_H\omega_m}{C^2}\boldsymbol{M}\right)\boldsymbol{J}=\boldsymbol{J}_{\boldsymbol{0}},$$

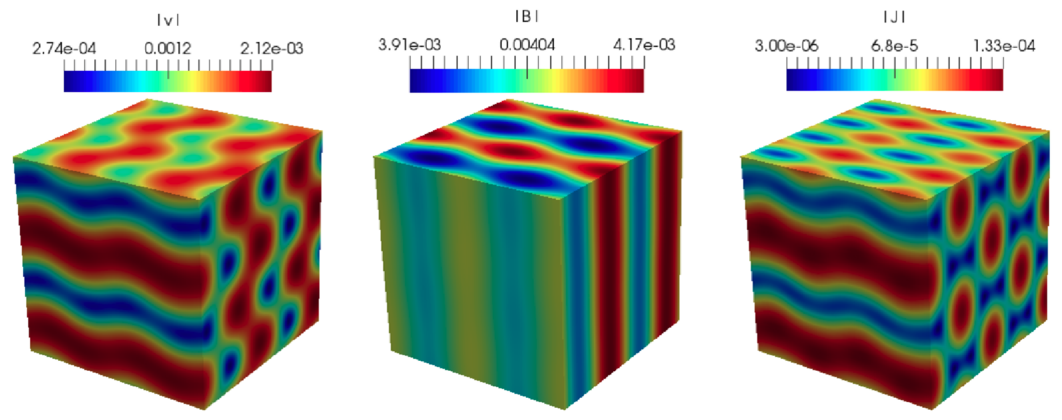
$$\boldsymbol{M}=\begin{pmatrix}0 & B_z & -B_y\\ -B_Z & 0 & B_x\\ B_y & -B_x & 0\end{pmatrix}, \qquad \boldsymbol{J}_{\boldsymbol{0}}=\begin{pmatrix}\Lambda_{yz}-\Lambda_{zy}-2(u_yB_z-u_zB_y)\\ \Lambda_{zx}-\Lambda_{xz}-2(u_zB_x-u_xB_z)\\ \Lambda_{xy}-\Lambda_{yx}-2(u_xB_y-u_yB_x)\end{pmatrix}$$

Features of the 3D Hall MHD LB scheme

- Multi-GPUs implementation allows high-resolution simulations with a short computational time: 512^3 run for t ~ $100T_0$ (T_0 integral time scale) in about 90 mins with 3xNVIDIA A100-40GB GPUs (CINECA, Italy).
- **Central-Moments** (**CMs**) scheme for the fluid *collision operator* increases the accuracy and stability of the scheme even at *high Reynolds number*.
- **D3Q7-BGK** implementation for the magnetic lattice preserves a significant amount of memory, while **D3Q27-CM** scheme for the fluid lattice ensures a high isotropy and accuracy of the algorithm.
- Fluid equilibrium distribution function expanded up to the *sixth-order in Hermite polynomials* increases the code stability and accuracy.

Code Validation

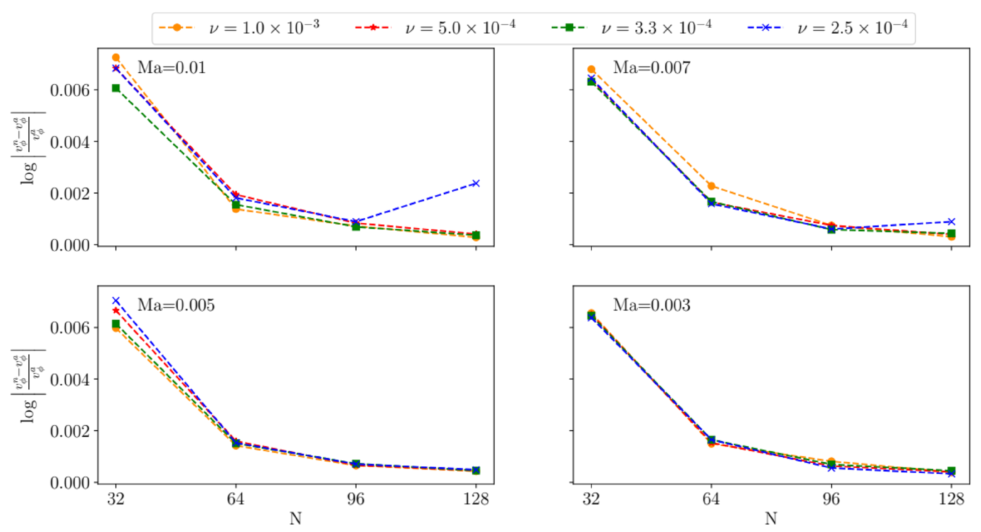
Validation of the code against the analytical wave-solution (Xia et al., PoP, 2015) for (non-linear) incompressible Hall MHD.



Validation consists in evaluating the **stability** and **accuracy** of our LB code in the parameter space (Mach number *Ma*, resolution *N*, and viscosity *ν*).

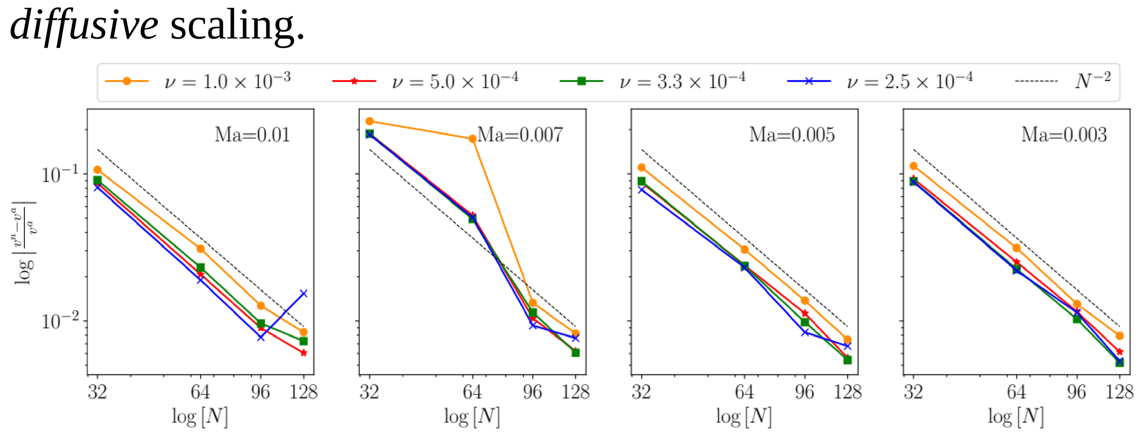
- **Stability**: the algorithm remains **numerically stable** (NO blowing up runs) for high Reynolds number. Nevertheless, simulations can transition to a turbulent regime (at low viscosity) due to non-linear dynamics.
- **Accuracy**: since we are dealing with waves, both the **dispersion** and **dissipation** errors are examined.

Dispersion error: it does not depend on the Mach number and/or kinematic viscosity, and it is significantly reduced by increasing the resolution *N*.

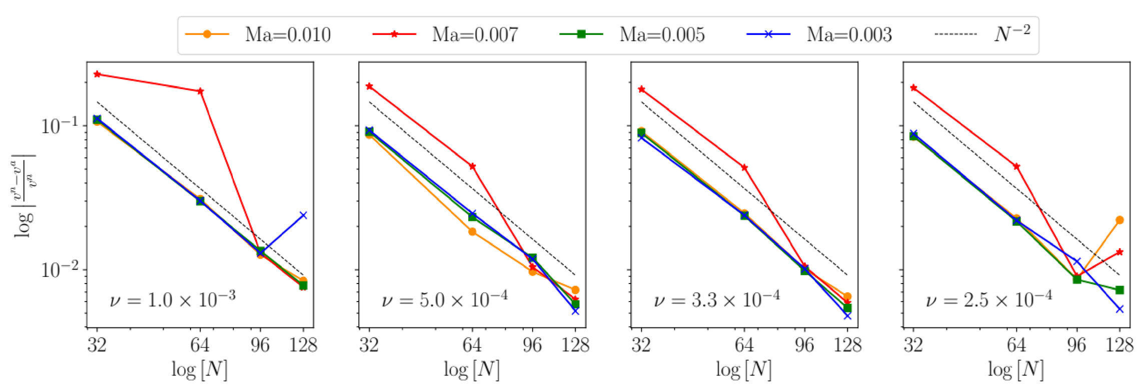


Dissipation error: it is evaluated in two different regimes, namely *acoustic* and *diffusive* scaling.

Acoustic Scaling



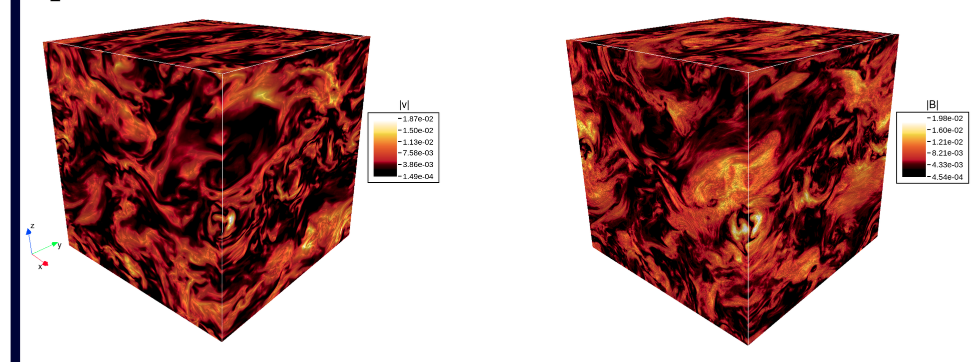
Diffusive Scaling

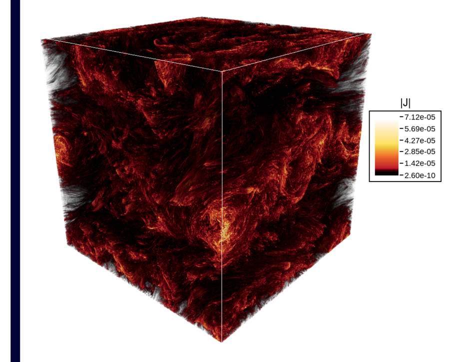


As expected, LB is second-order accurate in space and time $O(\Delta x^2,\Delta t^2)$.

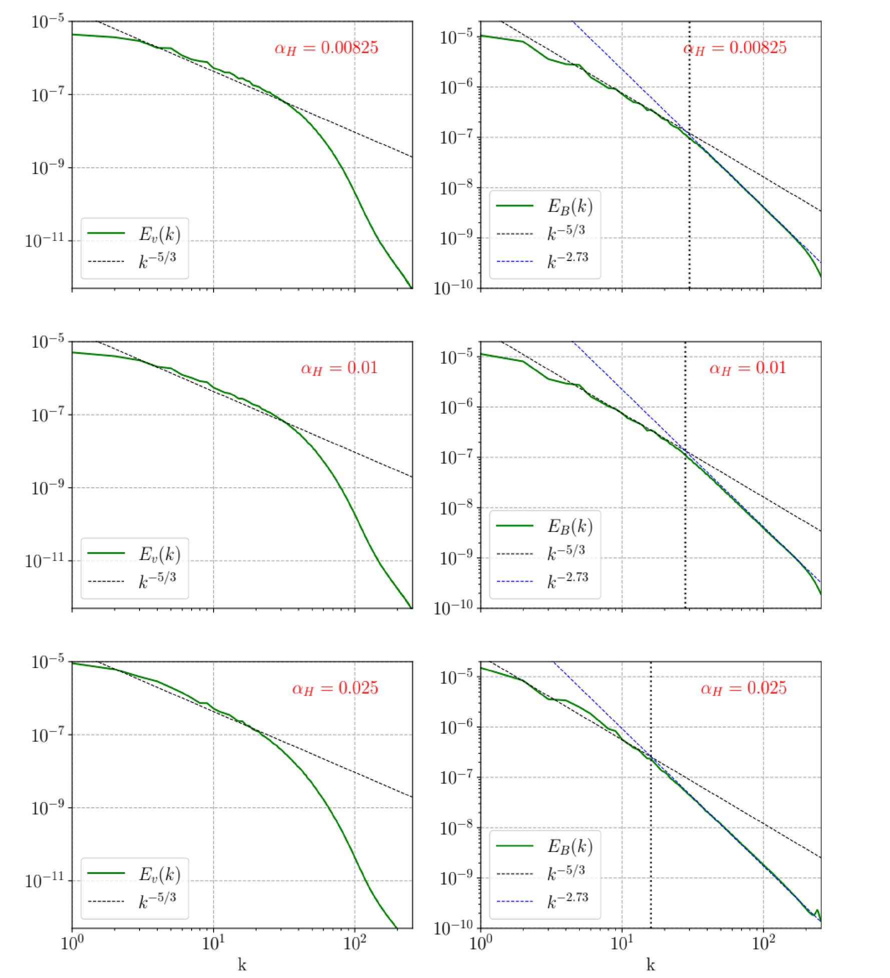
Hall MHD turbulence

Benchmark based on the Orszag-Tang vortex problem.





3D rendering of the **velocity** (*top left*), **magnetic field** (*top right*) and **current density** (*left*) at the peak of dissipation (t ~ $100T_0$) for a 512^3 run with Ma = 0.005, Re(t=0)=20000, α_H = 0.01, Pr_m=1.



Kinetic (left) and magnetic (right) spectra for three different runs with resolution N = 512.

- The magnetic spectra in the sub-ion range show a slope observed in solar wind measurements (Kiyani et al., Ph. Trans. R. Soc. A, 2015).
- The sub-ion range moves according to the intensity of the Hall parameter

References

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